# Discrete Mathematics



Computers represent everything using <u>bits</u> that can be either on (1) or off (0).

Because everything is stored as combinations of bits, computers can only deal with discrete data, as opposed to continuous data.

(Examples?)

To solve a problem by computer:

1.° Give a very precise specification of the problem. (what is the input?, what is the output?)

2.° Give a very precise description of how to solve the problem → programming.

3.° Argue that the program <u>correctly</u> solves the problem
(for every possible input, always terminates)

We'll focus on ~~building~~ tools ~~you need~~ that are useful for all steps:

Logic, Sets, Functions, Algorithms, Proofs, Graphs

# Logic



- Give a precise meaning to mathematical statements
- Combine ~~Combining~~ these statements into larger statements
- Argue ~~Arguing~~ that a complex mathematical statement is true (or false).

"building block" of logic: propositions

Proposition: statement that is true or false, but not both.

1. $1+1=2$ true
2. $5-2=2$ false
3. "Winter is coming" true

(Examples that aren't?)

We often ~~assign~~ use a variable name to denote a proposition

$p$ : "Jon Snow knows nothing."

Every proposition has a truth value:

T if it is true
F if it is false

# Combining Propositions



Negation: If $p$ is a proposition, then $\neg p$ is the proposition "It is not the case that $p$".

Truth value of $\neg p$ is the oppostie of $p$.

We can express this as a truth table:

| $p$ | $\neg p$ |
|---|---|
| T | F |
| F | T |

Example: $1+1=3$, Winter is coming

Conjunction: If $p$ and $q$ are propositions, then $p \wedge q$ is the proposition "$p$ and $q$".

Truth value of $p \wedge q$: $\begin{cases} T & \text{if both } p \text{ and } q \text{ are true} \\ F & \text{otherwise} \end{cases}$

Truth table:

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

Example: $1+1 \neq 3$ $\wedge$ Winter is coming (Others? False?)

Disjunction: If $p$ and $q$ are propositions, then $p \vee q$ is the proposition "$p$ or $q$".

Truth value of $p \vee q$: $\begin{cases} T & \text{if at least one of } p \text{ and } q \text{ is true} \\ F & \text{otherwise} \end{cases}$

Truth Table:

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

Examples: $10 \times 2 = 18$ or "I'm wearing a blue shirt"

~~[illegible]~~
"I drink coffee with milk" or "I drink coffee with sugar"

Exclusive or: $p \oplus q$: "either $p$ or $q$, but not both".

~~T.v.~~ $\begin{cases} T & \text{if exactly one of } p \text{ and } q \text{ is true} \\ F & \text{otherwise} \end{cases}$

| $p$ | $q$ | $p \oplus q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

In ~~#~~ natural language "or" can mean "exclusive or"
- ~~[illegible]~~ Steak comes with fries $\underset{\oplus}{\text{or}}$ salad. (Ex?)
- Before starting COMP 3804, you need to pass COMP 2804 or COMP 2402.

# Truth tables for compound propositions



$\neg(\neg p \vee q)$

| $p$ | $q$ | $\neg p$ | $\neg p \vee q$ | $\neg(\neg p \vee q)$ |
|---|---|---|---|---|
| T | T | F | T | F |
| T | F | F | F | T |
| F | T | T | T | F |
| F | F | T | T | F |

<u>Tautology</u>: proposition that is always true, regardless of the truth values of the atomic propositions.

<u>Contradiction</u>: proposition that is always false

<u>Contingency</u>: otherwise.

$\neg p \vee (p \vee q)$

| $p$ | $q$ | $\neg p$ | $p \vee q$ | $\neg p \vee (p \vee q)$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | T | T |
| F | T | T | T | T |
| F | F | T | F | T |

Propositions are <u>logically equivalent</u> if they have the same truth values for any combination of the truth values of their atomic propositions.
We write $p \equiv q$.

$p \equiv \neg\neg p$ Suppose we want to show that 6

| $p$ | $p$ |
|---|---|
| T | T |
| F | F |

| $p$ | $\neg p$ | $\neg\neg p$ |
|---|---|---|
| T | F | T |
| F | T | F |

We can do this by making truth tables for both and showing that their columns are identical.

(Something logically equiv. to $\oplus$?)

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

| $p$ | $q$ | $p \wedge q$ | $\neg(p \wedge q)$ | $\neg p$ | $\neg q$ | $\neg p \vee \neg q$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F |
| T | F | F | T | F | T | T |
| F | T | F | T | T | F | T |
| F | F | F | T | T | T | T |

same

# More connectives

**Implication:** $p \Rightarrow q$ : "If $p$, then $q$"

Truth value: $\begin{cases} F & \text{if } p \text{ is true, but } q \text{ is false} \\ T & \text{otherwise} \end{cases}$

| $p$ | $q$ | $p \Rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

Other ways to say it:
"$p$ implies $q$"
"$q$ follows from $p$"

**Examples:**

"Carleton is in Ottawa" $\Rightarrow 1+1=2$ - true!

$1+1=3 \Rightarrow$ "The moon is made of cheese" - true!

$\overset{p}{\text{"You get 100\% on the final"}} \Rightarrow \overset{q}{\text{"You get an A+"}}$

Suppose I tell you that this is true.

- you get 100%, I give you A+ } I kept my promise
- you get 100%, I don't give you A+ } I broke my promise, the implication is false!
- you do not get 100% – I may or may not give you an A+, either way, I kept my promise.

**Thus:** If $p$ is true, $q$ is definitely true too.
If $p$ is false, $q$ may or may not be true.

## If-and-only-if (bi-implication)

$p \Leftrightarrow q$: "p if and only if q" (p <u>iff</u> q)

$$\begin{cases} T & \text{if } p \text{ and } q \text{ have the same truth value} \\ F & \text{otherwise} \end{cases}$$

| $p$ | $q$ | $p \Leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

Ex: "The moon is made of cheese" $\Leftrightarrow$ $1+1=3$ - true

$1+1=2$ $\Leftrightarrow$ "Winter is coming" - true

"Carleton is in Ottawa" $\Leftrightarrow$ $1+1=3$ - false.

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Ex: $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

Ex: $p \Rightarrow q \equiv \neg p \vee q$

Ex: $p \wedge (p \vee q) \equiv p$