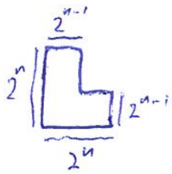
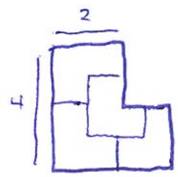
Example: We can tile any $2^n \times 2^n$ L-shape with L-trominoes, for $n \geq 1$.

L-Shape: 

L-tromino: 

Tiling: 

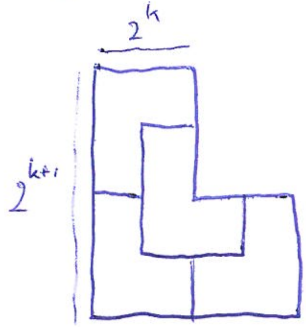
Proof: By induction on n.

Base Case ($n=1$): The 2×2 L-shape is an ~~are~~ L-tromino so it can easily be tiled.

Inductive Hypothesis: Assume that all $2^n \times 2^n$ L-shapes can be tiled, for $1 \leq n \leq k$.

Inductive Step: We need to show that the $2^{k+1} \times 2^{k+1}$ L-shape can be tiled by L-trominoes.

We can subdivide it ~~as follows:~~ into four $2^k \times 2^k$ L-shapes like this:



By our inductive hypothesis, each of the four smaller L-shapes can be tiled. Together, they form a tiling of the $2^{k+1} \times 2^{k+1}$ L-shape.

Thus, for $n \geq 1$, any $2^n \times 2^n$ L-shape can be tiled by L-trominoes.

<u>Example</u>: A long time ago, in a kingdom far, far away, they only have a \$1 coin and \$4 and \$5 bills. By Royal Decree, \$1 coins may only be used for amounts under \$12. Can the citizens still pay their bills?

<u>Answer</u>: Yes. Every amount of \$12 or over can be made with the \$4 and \$5 bills.

<u>Proof</u>: By induction on the amount.

<u>Base case</u>: $\$12 = 3 \times \$4$

$\$13 = \$5 + 2 \times \$4$

$\$14 = 2 \times \$5 + \$4$

$\$15 = 3 \times \$5$

<u>Inductive Hypothesis</u>: \$4 and \$5 bills can make any amount ~~up to \$k~~ between \$12 and \$k, for $\$k \geq \$15$.

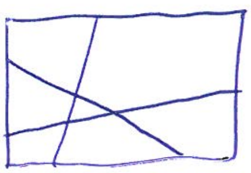
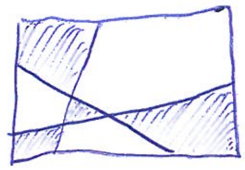
<u>Inductive Step</u>: We need to show that \$4 and \$5 bills can make $\$(k+1)$. ~~We know that $\$(k+1) \geq \$16$~~

$$\$(k+1) = \$4 + \$(k-3)$$

Since $\$k \geq \$15$, $k-3 \geq 12$. ~~By the~~ Thus we can apply the inductive hypothesis, which means that we can make $\$(k-3)$ with \$4 and \$5 bills. Then we can also make $\$(k+1)$ in that way~~, which~~

Therefore \$4 and \$5 bills can make any amount of \$12 and over.

Starting with a rectangle R, we can subdivide it with a few line segments:

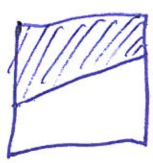
We can color this subdivision such that regions that 'touch' have different colours, and we only need two colours.

Does this work for <u>all</u> such subdivisions?

<u>Claim</u>: We can 2-colour any subdivision of a rectangle with ~~k~~ r lines, such that adjacent regions have different colours.

<u>Proof</u>: By induction on r.

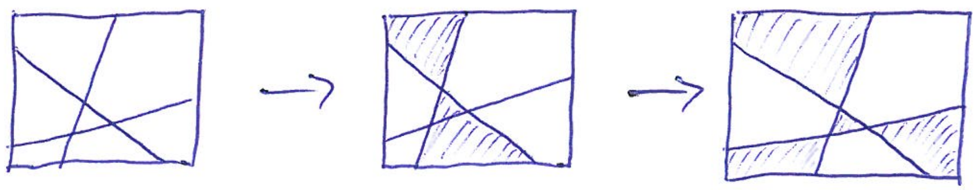
<u>Base Case</u> (r=1): We have only one segment, which splits the rectangle into two ~~parts~~ regions. Colour the first region black and the second white. This proves the base case.



<u>Inductive Hypothesis</u>: Assume that we can 2-colour any subdivision of a rectangle with r lines, for $1 \leq r \leq k$.

<u>Inductive Step</u>: We need to show that we can 2-colour a subdivision with k+1 lines.

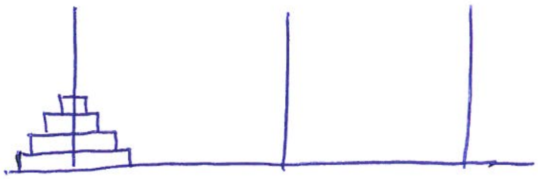
So suppose we have such a subdivision with $k+1$ lines. Pick a line ℓ and remove it. Now we have a subdivision with k lines, so by our inductive hypothesis, we can 2-color it. Now replace the removed line ℓ and note that it splits the rectangle into two parts, each of which is properly 2-coloured.



The only problem is that ~~the~~ two regions that touch along ℓ have the same colour. To fix this, we flip the colours of all regions on one side of ℓ. Now each part is still properly coloured and regions that touch along ℓ now have different colours, proving the inductive step. Therefore we can 2-colour any subdivision of a rectangle with lines.

~~Example~~

The Tower of Hanoi is a simple puzzle, made of 3 vertical ~~sticks~~ pegs and a number of discs ~~with a bo~~ of different sizes that can slide onto the pegs:



The goal is to move all discs to another peg, while following ~~the~~ three rules:

1. You can only move 1 disc at a time.
2. ~~In each~~ You can only move the top disc of each stack. ~~and you have to~~
3. ~~A~~ Bigger discs may not be placed on ~~a~~ smaller discs.

This puzzle becomes really easy if you use induction!

Example: We can ~~solve~~ move any ~~Tower of Hanoi puzzle with $n \geq 1$ discs.~~ stack of $n \geq 1$ discs to any other peg.

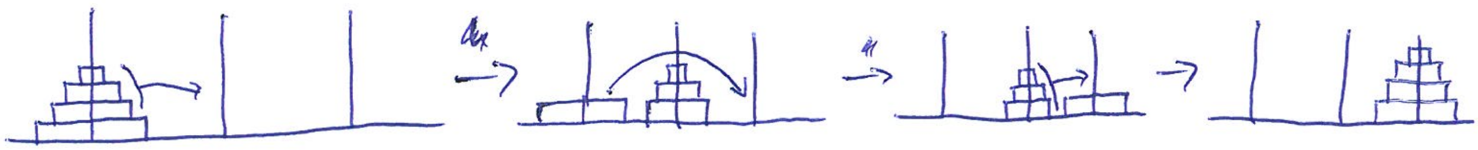
Proof: By induction on n.

Base Case ($n=1$): ~~If~~ since there is only 1 disc, we just move it to the correct peg.

Inductive Hypothesis: We can move any stack of $1 \leq n \leq k$ discs to any other peg.

Inductive Step: We have to show that we can move a stack of $k+1$ discs to another peg:

We can do this as follows:



we already know how to move a stack of k discs, so move the top k discs to the middle peg. Now the largest disc is free to move, so place it on the rightmost peg. Finally, move the k-stack from the middle peg onto the rightmost peg. ~~which pegs we use doesn't~~ Note that it doesn't really matter which pegs we use. Therefore we can move a stack of k+1 discs to any other peg, which means we can solve the Tower of Hanoi for any number of discs.

This shows that we can solve it (and gives an algorithm to do so!), but how many steps do we need?

If we write $s(n)$ for the number of steps to move a stack of n discs, we can deduce the following:

$$s(1) = 1$$

$$s(k+1) = 2 \cdot s(k) + 1$$

This is a <u>recursive function</u>.

Recursi~~me~~on

A sequence ~~a~~ or function is <u>recursive</u> if it is defined in terms of <u>itself</u>.

<u>Example</u>: ~~Let~~ Define the function $f: \mathbb{N} \to \mathbb{N}$ as

$$\begin{cases} f(0) = 3 \\ f(n+1) = 2 \cdot f(n) + 3 \quad (\text{for } n \geq 0) \end{cases}$$

$$f(1) = 2 \cdot f(0) + 3 = 2 \cdot 3 + 3 = 9$$
$$f(2) = 2 \cdot f(1) + 3 = 2 \cdot 9 + 3 = 21$$
$$\vdots$$

A recursive definition always has one or more <u>base cases</u> and one or more <u>recursive steps</u>.

<u>Example</u>: Give a recursive definition for ~~a~~
$$f(n) = n!$$

<u>Solution</u>:
$$\begin{cases} f(0) = 0! = 1 \\ f(n+1) = (n+1)! = 1 \cdot 2 \cdot \ldots \cdot n \cdot (n+1) = n! \cdot (n+1) \\ \qquad = f(n) \cdot (n+1) \quad (\text{for } n \geq 0) \end{cases}$$

<u>Exercise</u>: Give a recursive definition ~~of~~ for
$$f(n) = 5^n.$$

<u>Solution</u>:
$$\begin{cases} f(0) = 5^0 = 1. \\ f(n+1) = 5^{n+1} = 5 \cdot 5^n = 5 \cdot f(n) \quad (\text{for } n \geq 0) \end{cases}$$