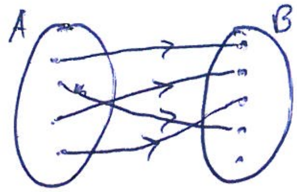
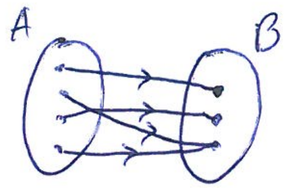
Countability

- Injective



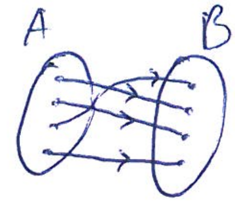
$|A| \leq |B|$

Surjective



$|A| \geq |B|$

Bijective



$|A| \leq |B| \wedge |A| \geq |B|$

$\equiv$

$|A| = |B|$

- What is the cardinality of $\{a, b, c, d\}$?

  Make a bijection:

  $a \to 1$

  $b \to 2$

  $c \to 3$

  $d \to 4$

  Therefore $|\{a, b, c, d\}| = |\{1, 2, 3, 4\}| = 4$

- Definition: Two sets A and B have the same cardinality if there exists a bijection from A to B.

- Definition: Set A is countable if
  - A is finite; or
  - $\{1, 2, 3, 4, \ldots\}$ has the same cardinality as A.

$\Rightarrow$ ~~Show:~~ To show that an infinite set A is countable, we need to give a bijection from $\{1,2,3,\dots\}$ to A.

Example: Show that $\overset{=\{0,1,2,\dots\}}{N}$ is countable.

~~Proof: Recall that $N=\{0,1,2,3,\dots\}$.~~

Method 1: ~~Give~~ Explicitly give the function.

$f:\{1,2,3,\dots\}\to N,\quad f(x)=x-1$

- f is injective: $f(x)=f(y)$

$$x-1=y-1$$
$$x=y$$

- f is surjective: $f(x)=b$

$$x-1=b$$
$$x=b+1$$

Therefore f is a bijection and N is countable.

Method 2: Find a way to list the elements of N, such that every element occurs exactly once.

$$\underset{f(1)}{0},\ \underset{f(2)}{1},\ \underset{f(3)}{2},\ \underset{f(4)}{3},\ \underset{f(5)}{4},\ \underset{f(6)}{5},\ \dots$$

This implicitly defines the bijection.

Exercise: Show that $\mathbb{Z}$ is countable.

Proof: List all elements of $\mathbb{Z}$ as follows:

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

Each integer occurs exactly once.

Proof 2: Let $f : \{1, 2, 3, \ldots\} \to \mathbb{Z}$ be defined as

$$f(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ -\frac{x-1}{2} & \text{if } x \text{ is odd} \end{cases}$$

f is a bijection.

Example: The set of odd positive integers is countable.

Proof: List the elements as follows:

$$1, 3, 5, 7, 9, \ldots$$

Each odd positive integer occurs exactly once.

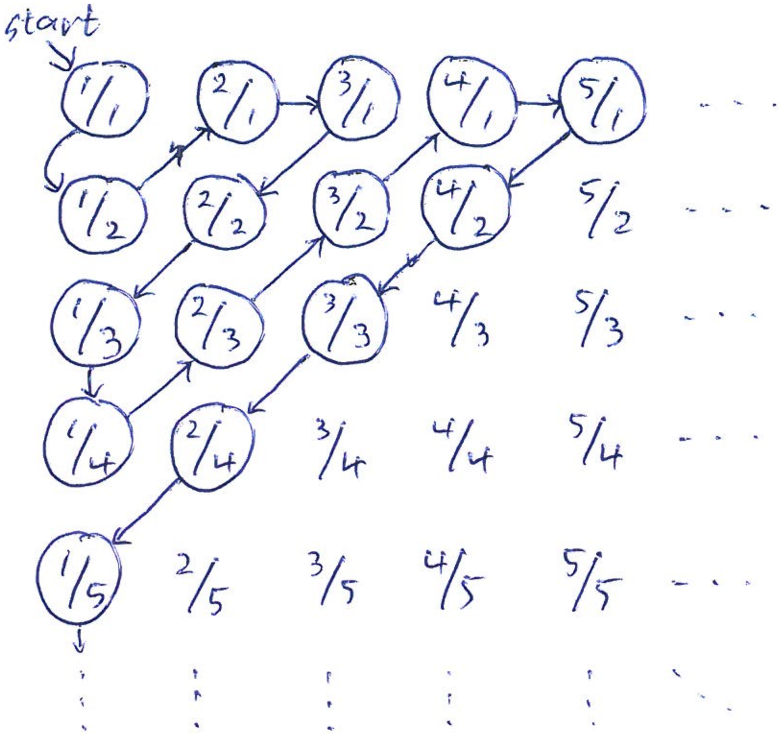
Proof 2: Let $f : \mathbb{N}^+ \to \{1, 3, 5, 7, 9, \ldots\}$ be defined as:

$$f(x) = 2x - 1$$

f is a bijection.

Example: $\mathbb{Q}^+ = \{p/q \mid p, q \in \mathbb{Z}^+\}$ is countable

Proof: Place the numbers on a grid, as follows.



This is a ~~as~~ sequence that includes every positive rational number <u>at least</u> once.

To get a list that ~~has~~ contains each element <u>exactly</u> once, write the numbers in this order, but skip fractions that can be simplified.

$$\underbrace{\tfrac{1}{1}}_{p+q=2},\ \underbrace{\tfrac{1}{2},\ \tfrac{2}{1}}_{p+q=3},\ \underbrace{\tfrac{3}{1},\ \tfrac{1}{3}}_{p+q=4},\ \underbrace{\tfrac{1}{4},\ \tfrac{2}{3},\ \tfrac{3}{2},\ \tfrac{4}{1}}_{p+q=5},\ \underbrace{\tfrac{5}{1},\ \tfrac{1}{5}}_{p+q=6},\ \dots$$

Therefore, there are as many positive rational numbers as positive integers!

<u>Claim:</u> $\mathbb{R}$ is <u>not</u> countable.

<u>Proof:</u> Consider just the real numbers between 0 and 1: $\{x \in \mathbb{R} \mid 0 < x < 1\}$.
Proof by contradiction. ~~If ... $\mathbb{R}$ is c~~
Assume that ~~$\mathbb{R}$ is coun~~ $\{x \in \mathbb{R} \mid 0 < x < 1\}$ is countable.
Then we can list <u>all</u> elements of this set:

$r_1 = 0.1957331\ldots$
$r_2 = 0.3881464\ldots$
$r_3 = 0.294513 2\ldots$
$r_4 = 0.5773809\ldots$
$\vdots$

Now define

$$r = 0.d_1 d_2 d_3 d_4 \cdots$$

where $d_i = \begin{cases} 4 & \text{if the } i\text{-th digit of } r_i \neq 4 \\ 7 & \text{otherwise} \end{cases}$

$$r = 0.4474\ldots$$

Clearly, r is a real number and $0 < r < 1$.
But r is not on the list, because

$$r \neq r_1 \wedge r \neq r_2 \wedge \ldots$$

In fact, $\forall_{n \in \mathbb{N}^+} (r \neq r_n)$.

Contradiction. ↯

Therefore, $\{x \in \mathbb{R} \mid 0 < x < 1\}$ is not countable.
Thus $\mathbb{R}$ is not countable. $\square$

To summarize:

- $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}^{(+)}$ are infinite sets, but they have the same size.
- $\mathbb{R}$ is also infinite, but is "much larger" than $\mathbb{N}$.

The set of all Java programs is countable: 78 42

Java program: finite string, whose symbols belong to the finite set $\{a, b, \ldots, z, A, B, \ldots, Z, 0, 1, \ldots, 9, =, <, >, ;, \ldots\}$

for each n: there are finitely many Java programs of length n.

How to list all possible Java programs:

* all programs of length 0
* all programs of length 1
* all programs of length 2

$\vdots$

Claim: The set of all functions $f: \mathbb{N} \to \mathbb{N}$ is uncountable.

Proof: By contradiction. Suppose it is countable, then we can list all functions:

$$f_1 = \{(0,1), (1,3), (2,5), \ldots$$
$$f_2 = \{(0,5), (1,4), (2,3), \ldots$$
$$f_3 = \{(0,2), (1,50), (2,25), \ldots$$

Now let $h: \mathbb{N} \to \mathbb{N}$ be defined as

$$h(x) = f_x(x) + 1$$

We have:

$h \neq f_1$, since $h(1) = f_1(1) + 1 \neq f_1(1)$

$h \neq f_2$, since $h(2) = f_2(2) + 1 \neq f_2(1)$

$\vdots$

Therefore h is not on the list.
This is a contradiction.
Hence, the set is uncountable.

So there ~~are~~ is a countably infinite number of Java programs, but an uncountably infinite number of functions on the natural numbers.

Therefore there are (lots of) functions that <u>cannot be computed!</u>