Example: $H(x) = $ "x is happy"

"All rich people are happy" $= \forall x \, (R(x) \Rightarrow H(x))$

"Not all rich people are happy"

$= \neg \, (\forall x \, (R(x) \Rightarrow H(x)))$

$\equiv \exists x \, (\neg \, (R(x) \Rightarrow H(x)))$

$\equiv \exists x \, (\neg \, (\neg R(x) \lor H(x)))$

$\equiv \exists x \, (\neg \neg R(x) \land \neg H(x))$

$\equiv \exists x \, (R(x) \land \neg H(x))$

$= $ "There exists ~~someone~~ a rich person who is not happy."

Nesting Quantifiers

Predicates can have more than one variable:

$P(x, y) = x < y$

In these cases, we often use nested quantifiers:

$\forall x \, (\exists y \, P(x, y)) = $ "For ~~all~~ each x, there exists a value y such that $x < y$."

$\underbrace{\qquad\qquad\qquad\qquad}$

true if $UoD = \mathbb{R}$, false if $UoD = \mathbb{Z}$

Note: $\forall x \exists y \, P(x, y) \not\equiv \exists y \forall x \, P(x, y)$

Order of the <u>same</u> quantifiers doesn't matter:

$\forall x \forall y \, P(x, y) \equiv \forall y \forall x \, P(x, y)$

Example: $\begin{cases} S(x) = \text{"x has a sister"} \\ K(x,y) = \text{"x knows y"} \end{cases}$ UoD = all people

"Everyone has a sister or knows someone who does."

$$= \forall x (S(x) \vee \exists y (S(y) \wedge K(x,y)))$$

Example: "There is exactly one rich person."

$R(x) = \text{"x is rich"}$ UoD = all people

$$\exists x (R(x) \wedge \forall y (R(y) \Rightarrow (x = y)))$$
$$(x \neq y) \Rightarrow \neg R(y)$$

Negating nested quantifiers

$\neg \forall x \exists y \, P(x,y)$

$\equiv \exists x (\neg \exists y \, P(x,y))$

$\equiv \exists x \forall y \, \neg P(x,y)$

Example: "Not everyone knows at least one person."
   $K(x,y) = \text{"x knows y"}$ UoD = all people

$$\neg \forall x \exists y \, K(x,y)$$
$$\equiv \exists x (\neg \exists y \, K(x,y))$$
$$\equiv \exists x \forall y \, \neg K(x,y)$$
$$= \text{"There is someone who doesn't know anybody."}$$

Validity of Logical Arguments

A logical argument:
- starts with a set of logical statements, called the <u>premise</u>.
- Ends with another logical statement - the <u>conclusion</u>

An argument is <u>valid</u> if the conclusion follows from the premise: if we assume that the premise is true, the conclusion <u>must</u> also be true.

Premise: $P_1, P_2, \ldots, P_k$ Conclusion: C

For a valid argument: $(P_1 \wedge P_2 \wedge \ldots \wedge P_k) \Rightarrow C$

An argument is <u>invalid</u> if it is possible for the conclusion to be false, when all premises are true.

<u>Example</u>: If you are at least 19, then you are allowed to drink beer. John is 21. Therefore, John is allowed to drink beer.

~~A(x)~~ $A(x) =$ "x is at least 19"
$B(x) =$ "x is allowed to drink beer."

Premise: $\forall x (A(x) \Rightarrow B(x)) \wedge A(\text{"John"})$ (UoD = all people)
Conclusion: $B(\text{"John"})$

- $(\forall x (A(x) \Rightarrow B(x))) \Rightarrow (A(\text{"John"}) \Rightarrow B(\text{"John"}))$ $\forall x P(x) \Rightarrow P(a)$
- $((A(\text{"John"}) \Rightarrow B(\text{"John"})) \wedge A(\text{"John"})) \Rightarrow B(\text{"John"})$ $((p \Rightarrow q) \wedge p) \Rightarrow q$

Therefore the argument is valid.

Example: All stupid kerbals are brave. Bill is not stupid. Therefore Bill is not brave.

$S(x) = $ "x is stupid" $UoD = $ all kerbals
$B(x) = $ "x is brave"

Premise: $\{\forall x \, (S(x) \Rightarrow B(x)) \wedge \neg S(\text{"Bill"})$
Conclusion: $\neg B(\text{"Bill"})$

Suppose that $S(x)$ is false for all kerbals, but $B(\text{"Bill"})$ is true. Then $\forall x \, (S(x) \Rightarrow B(x))$ is true, and $\neg S(\text{"Bill"})$ is true, but $\neg B(\text{"Bill"})$ is <u>false</u>.

Since awe found a case where all premises are true, but the conclusion is false, this argument is invalid.

Example: Gatineau Park is closed for biking, but it is sunny. If we go biking in GP then it is open for biking. If we don't go biking in GP, then we work on 1805. If we work on 1805, then we study for the test. Therefore we study for the test.

$P = $ "GP is open for biking"
$S = $ "It is sunny"
$B = $ "We go biking in GP"
$W = $ "We work on 1805"
$T = $ "We study for the test"

Premise: $- \neg P \wedge S$ $- \neg B \Rightarrow W$ Conclusion: T
$\quad\quad\quad\quad - B \Rightarrow P$ $- W \Rightarrow T$

- $\neg P \land S \Rightarrow \neg P$ $(p \land q \Rightarrow p)$
- $(\neg P \land (B \Rightarrow \neg P)) \Rightarrow \neg B$ $\{(p \land (q \Rightarrow \neg p)) \Rightarrow \neg p\}$
- $(\neg B \land (\neg B \Rightarrow W)) \Rightarrow W$ $\{(p \land (p \Rightarrow q)) \Rightarrow q\}$
- $(W \land (W \Rightarrow T)) \Rightarrow T$ "

Therefore the argument is valid.

Proofs: How to construct (valid) logical arguments.

Nearly all things we want (~~need~~) to prove are implications: $(P_1 \land P_2 \land \ldots \land P_k) \Rightarrow C$.

$$\underbrace{(P_1 \land P_2 \land \ldots \land P_k)}_{P} \Rightarrow C$$

A **Direct proof** assumes P and gives a sequence of valid steps that show that C is also true.

Example: If n is an odd integer, then n^2 is odd too.

Proof: Assume n is odd. Then we can write it as $n = 2k+1$, for some integer k.

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

Since $2k^2 + 2k$ is an integer, ~~als~~ we can write n^2 as $2k' + 1$ for some integer k'. Therefore n^2 is odd.

An **indirect proof** assumes $\neg C$ and shows $\neg P$. It is a direct proof of the contra-positive.

Example: If n^2 is odd then n is odd.

Direct: $n^2 = 2k+1$. $\sqrt{2k+1} = \ldots$?

Indirect: ~~there Assume that~~ Prove that if n is not odd, then n^2 is not odd. Assume that n is not odd. Then n is even, so $n=2k$ for some integer k.

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$$

Therefore n^2 is not odd. □

Proof by Contradiction ~~assumes $\neg(P\Rightarrow C) \equiv P \wedge \neg C$ and derives~~ assumes $\neg(P\Rightarrow C) \equiv P \wedge \neg C$ and derives False. ~~usually by showing that~~ It is a direct proof of $\neg(P\Rightarrow C) \Rightarrow F$, which is logically equivalent to $P \Rightarrow C$:

$$\neg(P\Rightarrow C) \Rightarrow F$$
$$\equiv \neg\neg(P\Rightarrow C) \vee F \quad \text{(Implication Equiv.)}$$
$$\equiv \neg\neg(P\Rightarrow C) \quad \text{(Identity)}$$
$$\equiv P\Rightarrow C \quad \text{(Double Negation)}$$

Example: If n is an integer and $3n+2$ is odd, then n is odd.

Proof: Assume that $3n+2$ is odd, but n is not odd. Then n is even and we can write $n=2k$ for some integer k.

$$3n+2 = 3(2k)+2 = 6k+2 = 2(3k+1)$$

Thus $3n+2$ is even. ↯

~~Therefore our assumption that n is not odd is false~~

Therefore n is ~~even~~ odd if $3n+2$ is odd.

Example: Prove that $\sqrt{2}$ is irrational.

Proof: Assume that $\sqrt{2}$ is rational.

Then we can write $\sqrt{2} = \frac{a}{b}$ for two integers a, b, and with $b \neq 0$, and such that a and b are not both even.

$$\sqrt{2} = \frac{a}{b}, \qquad 2 = \frac{a^2}{b^2}, \qquad a^2 = 2b^2$$

Therefore a^2 is even, which means that a is even. So we can write $a = 2k$, for some integer k. Then $a^2 = (2k)^2 = 4k^2 = 2b^2$, and $b^2 = 2k^2$. Therefore b^2 is even, meaning that b is also even. Thus, a and b are both even. ↯

Conclusion: $\sqrt{2}$ is not rational.

Example: If $n = ab$, where $a \geq 1$ and $b \geq 1$ are integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Proof: Assume $n = ab$, but $\neg(a \leq \sqrt{n} \vee b \leq \sqrt{n})$. So $a > \sqrt{n}$ and $b > \sqrt{n}$. But then

$$n = ab > \sqrt{n} \cdot \sqrt{n} = n. \quad ↯$$

If we need to prove $(P_1 \vee P_2 \vee \ldots \vee P_k) \Rightarrow C$, we can instead prove $(P_1 \Rightarrow C) \wedge (P_2 \Rightarrow C) \wedge \ldots \wedge (P_k \Rightarrow C)$. This is a proof by cases.