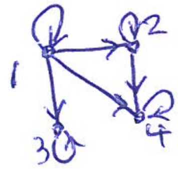
Relations

A <u>relation</u> on a set A is a subset of $A \times A$.
We can represent it ~~wi~~ as a directed graph:

<u>Example</u>: $A = \{1, 2, 3, 4\}$. $R_0 = \{(a,b) \mid a, b \in A \wedge a \text{ divides } b\}$

$$R_0 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$$



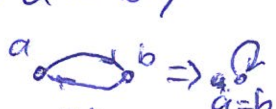
~~Definition~~

A relation R is:

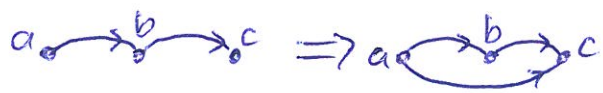
- <u>Reflexive</u> if $\forall a \in A\,((a,a) \in R)$

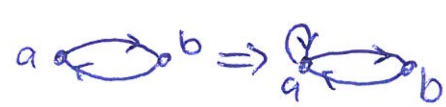
- <u>Symmetric</u> if $\forall a, b \in A\,((a,b) \in R \Rightarrow (b,a) \in R)$

- <u>Anti-symmetric</u> if $\forall a, b \in A\,((a,b) \in R \wedge (b,a) \in R \Rightarrow a = b)$



- <u>Transitive</u> if $\forall a, b, c \in A\,((a,b) \in R \wedge (b,c) \in R \Rightarrow (a,c) \in R)$





Note: anti-symmetric $\neq$ not symmetric.

Example: $R_1 = \{(a,b) \mid a=b\}$, ~~R~~

- R_1 is reflexive: $a=a$
- R_1 is symmetric: $a=b \Rightarrow b=a$
- R_1 is anti-symmetric: $a=b \wedge b=a \Rightarrow a=b$
- R_1 is transitive: $a=b \wedge b=c \Rightarrow a=c$

Example: $R_2 = \{(a,b) \mid a+b \leq 3\}$

- R_2 is not reflexive: $(2,2) \notin R_2$
- R_2 is symmetric: $a+b \leq 3 \Rightarrow b+a \leq 3$
- R_2 is not anti-symmetric: $1+2 \leq 3 \wedge 2+1 \leq 3$, but $1 \neq 2$.
- R_2 is not transitive: $2+1 \leq 3 \wedge 1+2 \leq 3$, but $2+2 \overset{>}{\cancel{\leq}} 3$.

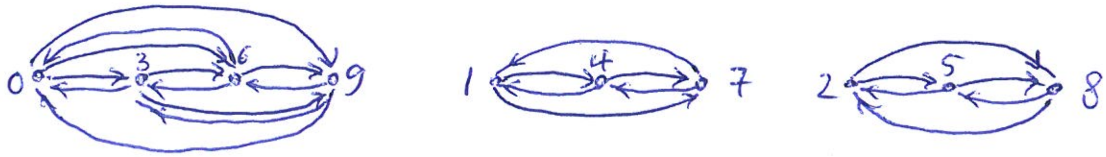
A relation is an <u>equivalence relation</u> if it is reflexive, symmetric, and transitive.

Example: $R_3 = \{(a,b) \mid a-b \text{ is divisible by } 3\}$ $A=\{0,\ldots,\overset{9}{\cancel{10}}\}$

- R_3 is reflexive: $a-a=0$ is divisible by 3
- R_3 is symmetric: if $a-b = 3k$, then $b-a = 3\cdot(-k)$.
- R_3 is transitive:

$$\begin{array}{r} a-b = 3\cdot k \\ b-c = 3\cdot \ell \; + \\ \hline a-c = 3\cdot(k+\ell) \end{array}$$

Drawing R_3:



An equivalence relation partitions A into <u>equivalence classes</u>:

$$[a]_R = \{b \mid (a,b) \in R\} \quad \text{equivalence class of } a.$$

<u>Example</u>: $[0]_R = \{b \mid 0-b \text{ is divisible by } 3\}$
$= \{0, 3, 6, 9\}$
$= [3]_R = [6]_R = [9]_R$

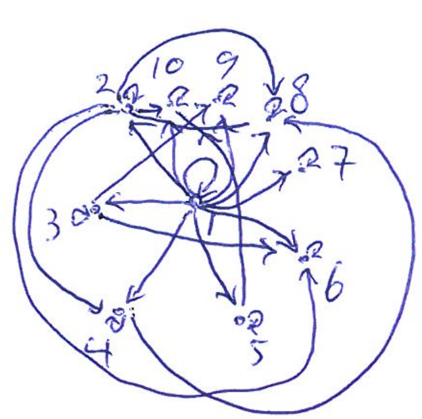
- All elements in the same equivalence class are related.
- Elements from different equivalence classes are not related.

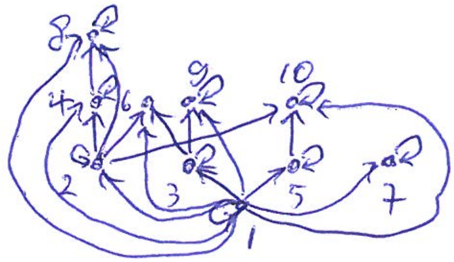
A relation is a <u>partial order</u> if it is reflexive, anti-symmetric, and transitive.

<u>Example</u>: $R_4 = \{(a,b) \mid a \text{ divides } b\}$ $A = \{1, \dots, 10\}$

- R_4 is reflexive: a divides a
- R_4 is anti-symmetric: if a divides b and b divides a, then $a = b$.
- R_4 is transitive: if a divides b and b divides c, then a divides c.

Drawing R_4:



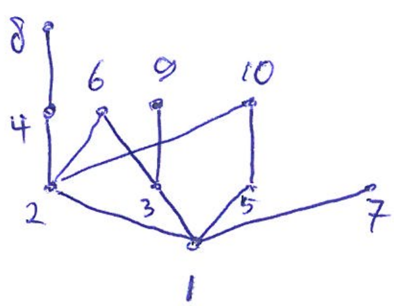


A cleaner way (<u>Hasse diagram</u>):

1) Remove all loops (just remember they're there)

2) Indicate direction by drawing the end point higher

3) Remove (but remember) transitive edges.



An element a is <u>maximal</u> if there is no strictly 'higher' element: $\forall b \in A\,(R(a,b) \Rightarrow a = b)$

Similarly, an element a is <u>minimal</u> if there is no strictly 'lower' element: $\forall b \in A\,(R(b,a) \Rightarrow a = b)$.

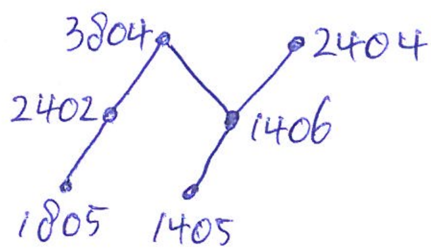
In R4, 6, 7, 8, 9, and 10 are all maximal.
Only 1 is minimal.
If there is only one minimal or maximal element, this element is the <u>minimum</u> or <u>maximum</u>.

In R4, 1 is the minimum, but there is no maximum.

Topological Sort

It is often useful to find a total order that ~~is com~~ satisfies the constraints of a given partial order.

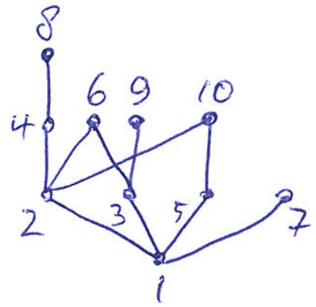
<u>Example</u>: Course pre-reqs:



Valid order for courses:
~~Take courses in order~~ 1805, 2402, 1405, 1406, 2404, 3804 ~~(for example)~~.

<u>Topological Sort</u>: Remove a minimal element and place it at the next spot in the total order. Repeat until all elements are ordered.

Example:



Total order: 1, 7, 3, 9, 5, 2, 4, 6, 8, 10.

Also valid: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

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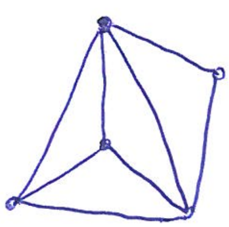
Planarity

A graph is *planar* if it can be drawn with: (start with 88-93)

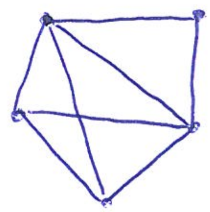
- vertices as points
- edges as straight line segments connecting their endpoints
- no edge crossings.

Such a drawing is called *plane*.

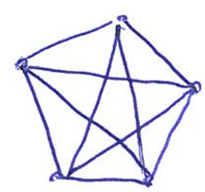
Example:



Plane drawing of a planar graph

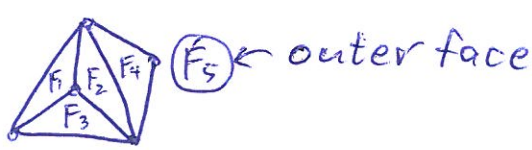


Non-plane drawing of a planar graph



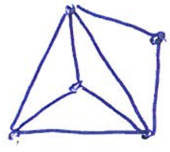
Non-planar graph

A plane graph partitions the plane into *faces*:



Euler's Formula: In a connected plane graph $G=(V,E)$ with faces F, $|V|-|E|+|F|=2$.

Example:



5 faces
5 vertices
8 edges

$5-8+5=10-8=2$ ✓

Proof: By induction on $|E|$. (87)

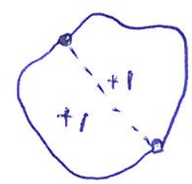
Base Case ($|E| = 0$): $G = \bullet$. $|V| - |E| + |F| = 1 - 0 + 1 = 2$ ✓

Inductive Hypothesis: ~~In~~ Any connected plane graph with $0 \le |E| \le k$ ~~edges~~, $|V| - |E| + |F| = 2$.

Inductive Step: Let G be a ~~graph~~ connected plane graph with $k+1$ edges. ~~Delete Remove one edge e~~ Let G' be G with one edge e removed. There are two cases:

1. G' is connected. By the IH, $|V'| - |E'| + |F'| = 2$.

   ~~$|V'| = |V|$, $|E'| = |E| - 1$, $|F'$~~

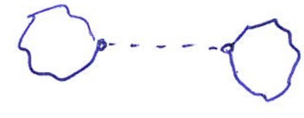


Then
$$|V| - (|E| - 1) + (|F| - 1) = 2$$
$$|V| - |E| + 1 + |F| - 1 = 2$$
$$|V| - |E| + |F| = 2.$$

2. G' is disconnected. Then it has two parts G'_1 and G'_2 that are connected. By the IH,

$$|V'_1| - |E'_1| + |F'_1| = 2$$ ~~and $|V'_2| - |E'_2| + |F'_2| = 2$~~

and
$$\underline{|V'_2| - |E'_2| + |F'_2| = 2} \; +$$



$$(|V'_1| + |V'_2|) - (|E'_1| + |E'_2|) + (|F'_1| + |F'_2|) = 4$$
~~$|V'| - |E'| + (|F'| + 1) = 4$~~
$$|V| - (|E| - 1) + (|F'| + 1) = 4$$
$$|V| - |E| + |F| + 2 = 4$$
$$|V| - |E| + |F| = 2.$$

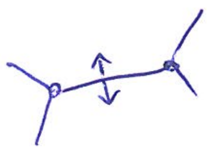
(not covered)

Claim: A planar graph with $|V| \geq 3$ has no more than $3|V|-6$ edges.

Proof:
- Euler: $|V| - |E| + |F| = 2$

$$\Rightarrow |E| = |V| + |F| - 2$$

- Every edge is incident on (at most) two faces:



$$2 \cdot |E| \geq \sum_{f \in F} |f|$$

- Every face has at least three edges:

$$2 \cdot |E| \geq \sum_{f \in F} |f| \geq 3 \cdot |F|$$

$$\Rightarrow |F| \leq \tfrac{2}{3}|E|$$

- Combining:

$$|E| = |V| + |F| - 2$$

$$|E| \leq |V| + \tfrac{2}{3}|E| - 2$$

$$\tfrac{1}{3}|E| \leq |V| - 2$$

$$|E| \leq 3|V| - 6$$