Tips for proving stuff

① Write down the statement (premises and conclusion) as precisely as possible.

② Simplify both parts as much as possible by applying definitions and identities.

③ Try applying any rules you know, to get from the premises to the conclusion.

④ If nothing works, repeat 1–3 for the contrapositive (indirect proof)

⑤ If still nothing works, repeat 1–3 for the negation $(P \wedge \neg C)$ and try to get a contradiction

Note: If the goal is to prove or disprove, switch between proving and finding a counter-example whenever you get stuck.

Example: Let f and g be two functions. Prove that if f and g are both surjective, $f \circ g$ is also surjective.

Proof: Suppose g maps A to B and f maps B to C. Then we know that $\forall b \in B(\exists a \in A(g(a)=b))$ and $\forall c \in C(\exists b \in B(f(b)=c))$. We need to prove that $\forall c \in C(\exists a \in A((f \circ g)(a)=c))$. This is equivalent to $\forall c \in C(\exists a \in A(f(g(a))=c))$. Let c be an arbitrary element of C. We need to find some $a \in A$ such that $f(g(a))=c$.

Since $c \in C$ and f is surjective, $\exists b \in B(f(b)=c)$. Let $b \in B$ be such ~~that~~ an element with $f(b)=c$.

Since $b \in B$ and g is surjective, $\exists a \in A(g(a)=b)$. ~~Con~~ Let $a \in A$ be such an element with $g(a)=b$.

Then $f(g(a)) = f(b) = c$.

Therefore, there exists an $a \in A$ such that $(f \circ g)(a) = c$

Thus, $f \circ g$ is surjective. $\square$

Sequences and Sums

Examples of a sequence: $\underset{a_1}{1}, \underset{a_2}{\frac{1}{2}}, \underset{a_3}{\frac{1}{3}}, \underset{a_4}{\frac{1}{4}}, \underset{a_5}{\frac{1}{5}}, \ldots$

Formally, a sequence is a function $a: \mathbb{N}^+ \to S$

~~We can describe this a sequence as $\{a_n\}$, where~~

Elements are written as a_i or $a(i)$.

The above sequence is $a: \mathbb{N}^+ \to \mathbb{Q}$, ~~$a(i) = \frac{1}{i}$~~. $a_i = \frac{1}{i}$

The summation of $a_m, a_{m+1}, \ldots, a_n$ is written as

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \ldots + a_{n-1} + a_n$$

Think of it as this program:

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for i=m to n
    sum = sum + a_i
return sum
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Example: $\sum_{i=1}^{15} 1 = \underset{a_1}{1} + \underset{a_2}{1} + \underset{a_3}{1} + \ldots + \underset{a_{15}}{1} = 15 \times 1 = 15$

Example: $\sum_{i=2}^{5} i^2 = 2^2 + 3^2 + 4^2 + 5^2$

$= 4 + 9 + 16 + 25$

$= 54$

Example: $\sum_{i=1}^{7} i = 1 + 2 + 3 + 4 + 5 + 6 + 7$

$= 28$

What is $\sum_{i=1}^{n} i$? Let's call it S.

$$S = 1 + 2 + 3 + \dots + (n-2) + (n-1) + n$$

$$S = n + (n-1) + (n-2) + \dots + 3 + 2 + 1 \quad +$$

$$2S = \underbrace{(n+1) + (n+1) + (n+1) + \dots + (n+1) + (n+1) + (n+1)}_{n}$$

$$= n * (n+1)$$

$$\therefore S = \frac{n(n+1)}{2}$$

Special Sums

- $\sum_{i=1}^{n} 1 = n$
- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$
- $\sum_{i=0}^{n} a \cdot r^i = \begin{cases} \frac{a \cdot r^{n+1} - a}{r-1} & \text{if } r \neq 1 \\ (n+1) \cdot a & \text{if } r = 1 \end{cases}$

Manipulating Sums

① Rename the variable: $\sum_{i=m}^{n} a_i = \sum_{k=m}^{n} a_k$

② Split the ~~sum~~ domain: $\sum_{i=m}^{n} a_i = \sum_{i=m}^{k-1} a_i + \sum_{i=k}^{n} a_i$ for $m \leq k \leq n$.

Example: $\sum_{i=1}^{7} i = \underbrace{1+2+3+4}+\underbrace{5+6+7}$

$$= \sum_{i=1}^{4} i + \sum_{i=5}^{7} i \implies \sum_{i=5}^{7} i = \sum_{i=1}^{7} i - \sum_{i=1}^{4} i$$

Example: $\sum_{i=6}^{n} i = \sum_{i=1}^{n} i - \sum_{i=1}^{5} i$

$$= \frac{n(n+1)}{2} - \frac{5(5+1)}{2}$$

$$= \tfrac{1}{2}n^2 + \tfrac{1}{2}n - 15$$

③ Split over addition: $\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$

Example: $\sum_{i=1}^{4} (2^i + i) = (2^1+1)+(2^2+2)+(2^3+3)+(2^4+4)$

$$= (2^1+2^2+2^3+2^4)+(1+2+3+4)$$

$$= \sum_{i=1}^{4} 2^i + \sum_{i=1}^{4} i$$

④ Extract common factors: $\sum_{i=m}^{n} c \cdot a_i = c \cdot \sum_{i=m}^{n} a_i$

Example: $\sum_{i=1}^{3} 4i = 4\cdot 1 + 4\cdot 2 + 4 \cdot 3$

$$= 4(1+2+3)$$

$$= 4 \cdot \sum_{i=1}^{3} i$$

<u>Example</u>: $\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}\left(i\times\sum_{j=1}^{3} j\right)$

$$= \sum_{i=1}^{4}(i\times 6)$$

$$= 6\times\sum_{i=1}^{4} i$$

$$= 6\times 10$$

$$= 60$$

⑤ ~~Replace the variable by an expression:~~ Shift the domain:

$\cancel{\sum_{i=m}^{n} a_i = \sum_{f(k)=m}^{f(k)=n} a_{f(k)}}$ ~~Special case~~ $\sum_{i=m}^{n} a_i = \sum_{j=m+x}^{n+x} a_{i-x}$ $(x\in\mathbb{Z})$

<u>Example</u>: $\sum_{i=1}^{3} 2^{i-1} = 2^0+2^1+2^2 = \cancel{2+4+8=14}$

$1+2+4=7$

$\cancel{\sum_{i=1}^{3} 2^i = \sum_{i=0}^{3} 2^i - \sum_{i=0}^{0} 2^i = 2^{3+1}-1-(2^{0+1}-1) = 16-1-(2-1) = 14}$

$$\sum_{i=1}^{3} 2^{i-1} = \sum_{j=1-1}^{3-1} 2^{(j+1)-1} = \sum_{j=0}^{2} 2^{j} = 2^{2+1}-1 = 7$$

<u>Example</u>: Show that for $r\neq 1$, $\sum_{i=0}^{n} a\cdot r^i = \frac{a\cdot r^{n+1}-a}{r-1}$.

<u>Proof</u>: Let $S=\sum_{i=0}^{n} a\cdot r^i$. Then

$$r\cdot S = r\cdot\sum_{i=0}^{n} a\cdot r^i$$

$$= \sum_{i=0}^{n} a\cdot r^{i+1}$$

$$= \sum_{j=1}^{n+1} a\cdot r^j$$

$$= \sum_{j=0}^{n} a\cdot r^j - \sum_{j=0}^{0} a\cdot r^j + \sum_{j=n+1}^{n+1} a\cdot r^j$$

$$= \underset{S}{\cancel{\sum_{i=0}^{n} a \cdot r^{i}}} - a \cdot r^{0} + a \cdot r^{n+1}$$

$$= S + a \cdot r^{n+1} - a$$

$$rS - S = a \cdot r^{n+1} - a$$

$$(r-1)S = a \cdot r^{n+1} - a$$

$$S = \frac{a \cdot r^{n+1} - a}{r-1} \qquad \square$$

Evaluating Sums

Example: $\displaystyle\sum_{i=1}^{n} (7i-6) = \sum_{i=1}^{n} 7i - \sum_{i=1}^{n} 6$

$$= 7 \cdot \sum_{i=1}^{n} i - 6 \cdot \sum_{i=1}^{n} 1$$

$$= 7 \cdot \frac{n(n+1)}{2} - 6n$$

Exercise: $\displaystyle\sum_{i=1}^{n} (i^2+3i-4) = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} 3i - \sum_{i=1}^{n} 4$

$$= \sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i - 4\sum_{i=1}^{n} 1$$

$$= \frac{n(n+1)(2n+1)}{6} + 3 \cdot \frac{n(n+1)}{2} - 4n$$

Exercise: $\displaystyle\sum_{i=1}^{n} \sum_{j=3}^{n} ij = \sum_{i=1}^{n} \left(i \cdot \sum_{j=3}^{n} j \right)$

$$= \sum_{i=1}^{n} \left(i \cdot \left(\sum_{j=1}^{n} j - \sum_{j=1}^{2} j \right) \right)$$

$$= \sum_{i=1}^{n} \left(i \cdot \left(\frac{n(n+1)}{2} - 3 \right) \right)$$

$$= \left(\frac{n(n+1)}{2} - 3 \right) \cdot \sum_{i=1}^{n} i$$

$$= \left(\frac{n(n+1)}{2} - 3 \right) \cdot \frac{n(n+1)}{2}$$

Example:

$$\sum_{i=1}^{n}\sum_{j=i}^{n}(j-i)=\sum_{i=1}^{n}\left(\sum_{j=i}^{n} j-\sum_{j=i}^{n} i\right)$$

$$=\sum_{i=1}^{n}\left(\sum_{j=1}^{n} j-\sum_{j=1}^{i-1} j-\left(\sum_{j=1}^{n} i-\sum_{j=1}^{i-1} i\right)\right)$$

$$=\sum_{i=1}^{n}\left(\frac{n(n+1)}{2}-\frac{(i-1)i}{2}-\left(i\cdot\sum_{j=1}^{n} 1-i\cdot\sum_{j=1}^{i-1} 1\right)\right)$$

$$=\sum_{i=1}^{n}\left(\frac{n(n+1)-(i-1)i}{2}-i\cdot\left(n-(i-1)\right)\right)$$

$$=\sum_{i=1}^{n}\left(\frac{n(n+1)}{2}-\frac{(i-1)i}{2}-in+i(i-1)\right)$$

$$=\sum_{i=1}^{n}\left(\frac{n(n+1)}{2}+\frac{(i-1)i}{2}-in\right)$$

$$=\sum_{i=1}^{n}\frac{n(n+1)}{2}+\sum_{i=1}^{n}\frac{(i-1)i}{2}-\sum_{i=1}^{n} in$$

$$=\cancel{\sum_{i=1}^{n}}\frac{n(n+1)}{2}\cdot\sum_{i=1}^{n} 1+\frac{1}{2}\sum_{i=1}^{n}(i^2-i)-n\cdot\sum_{i=1}^{n} i$$

$$=\frac{n(n+1)}{2}\cdot\sum_{i=1}^{n} 1+\frac{1}{2}\sum_{i=1}^{n} i^2-\frac{1}{2}\sum_{i=1}^{n} i-n\cdot\sum_{i=1}^{n} i$$

$$=\frac{n(n+1)}{2}\cdot n+\frac{1}{2}\cdot\frac{n(n+1)(2n+1)}{6}-\frac{1}{2}\cdot\frac{n(n+1)}{2}-n\cdot\frac{n(n+1)}{2}$$

$$=\frac{1}{2}\cdot\left(\frac{n(n+1)(2n+1)}{6}-\frac{3n(n+1)}{6}\right)$$

$$=\frac{n(n+1)(2n-2)}{12}$$

End of L7