(Recap: · truth tables for $\wedge, \vee, \Rightarrow, \Leftrightarrow$) right board ⑨ (L2)

· logical equivalence

Assignment due in class.

Two propositions are <u>logically equivalent</u> if they always have the same truth value. Alternatively, ~~xxx~~ $p \Leftrightarrow q$ is a tautology.

This is useful for simplifying logical statement.

Ex: We know that $p \wedge (p \vee q) \equiv p$.

So then $(p \wedge (p \vee q)) \Rightarrow (q \wedge (q \vee p)) \equiv p \Rightarrow q$.

There are two ways to show that two propositions are logically equivalent:

1) Make a truth table for both and compare the last columns

2) Rewrite one into the other using a series of <u>known</u> logical equivalences.

<u>Logical Equivalences</u> (left board)

$p \wedge T \equiv p$	$p \vee F \equiv p$	(Identity Law)
$p \wedge F \equiv F$	$p \vee T \equiv T$	(Domination Law)
$p \wedge p \equiv p$	$p \vee p \equiv p$	(Idempotent Law)
$p \wedge \neg p \equiv F$	$p \vee \neg p \equiv T$	(Negation Law)
$\neg\neg p \equiv p$		(Double Negation Law)
$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$	(Commutative Law)
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$		(Associative Law)

$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ (Distributive Law)

(Ex: verify with a truth table)

$\neg(p \land q) \equiv \neg p \lor \neg q$ $\quad \neg(p \lor q) \equiv \neg p \land \neg p$ (De Morgan's Law)

(Example: while(a ∨ b) do ... end. what do we know after the loop?)

$p \land (p \lor q) \equiv p$ $\quad p \lor (p \land q) \equiv p$ (Absorption Law)

(Ex: show by using previous rules. Tip: $(p \land T) \lor (p \land q)$)

$p \Rightarrow q \equiv \neg p \lor q$ (Implication Equivalence)

$p \Leftrightarrow q \equiv (p \Rightarrow q) \land (q \Rightarrow p)$ (Bi-implication Equivalence)

(Contrapositive)

$\neg p \Rightarrow \neg q \equiv q \Rightarrow p$

Example: Show that $\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg q$.

Solution 1: Use a truth table.

Solution 2:

$$
\begin{aligned}
\neg(p \lor (\neg p \land q)) &\equiv \neg((p \lor \neg p) \land (p \lor q)) && \text{(Distributive Law)}\\
&\equiv \neg(T \land (p \lor q)) && \text{(Negation Law)}\\
&\equiv \neg(p \lor q) && \text{(Identity Law)}\\
&\equiv \neg p \land \neg q && \text{(De Morgan's Law)}
\end{aligned}
$$

(Alternative: De Morgan first)

Example: Show that $(p \land q) \Rightarrow (p \lor q)$ is a tautology.

$$
\begin{aligned}
(p \land q) \Rightarrow (p \lor q) &\equiv \neg(p \land q) \lor (p \lor q) && \text{(Implication)}\\
&\equiv (\neg p \lor \neg q) \lor (p \lor q) && \text{(De Morgan)}\\
&\equiv (\neg p \lor p) \lor (\neg q \lor q) && \text{(Commutivity and Associativity)}\\
&\equiv T \lor T && \text{(Negation)}\\
&\equiv T && \text{(Domination)}
\end{aligned}
$$

Rules of Thumb

① Remove all bi-implications and implications.

Ex: $(A \Rightarrow B) \Leftrightarrow (\neg B \wedge A)$

$\equiv ((A \Rightarrow B) \Rightarrow (\neg B \wedge A)) \wedge ((\neg B \wedge A) \Rightarrow (A \Rightarrow B))$

$\equiv (\neg (A \Rightarrow B) \vee (\neg B \wedge A)) \wedge (\neg(\neg B \wedge A) \vee (A \Rightarrow B))$

$\equiv (\neg (\neg A \vee B) \vee (\neg B \wedge A)) \wedge (\neg(\neg B \wedge A) \vee (\neg A \vee B))$

② Apply De Morgan as ~~long~~ many times as possible.

$(\neg(\neg A \vee B) \vee (\neg B \wedge A)) \wedge (\neg(\neg B \wedge A) \vee (\neg A \vee B))$

$\equiv ((\neg\neg A \wedge \neg B) \vee (\neg B \wedge A)) \wedge ((\neg\neg B \vee \neg A) \vee (\neg A \vee B))$

(Then use the other rules)

$\equiv ((A \wedge \neg B) \vee (\neg B \wedge A)) \wedge ((B \vee \neg A) \vee (\neg A \vee B))$

$\equiv ((A \wedge \neg B) \vee (A \wedge \neg B)) \wedge ((\neg A \vee B) \vee (\neg A \vee B))$

$\equiv (A \wedge \neg B) \wedge (\neg A \vee B)$

$\equiv A \wedge (\neg B \wedge (\neg A \vee B))$

$\equiv A \wedge ((\neg B \wedge \neg A) \vee (\neg B \wedge B))$

$\equiv A \wedge ((\neg B \wedge \neg A) \vee F)$

$\equiv A \wedge (\neg B \wedge \neg A)$

$\equiv (A \wedge \neg A) \wedge \neg B$

$\equiv F \wedge \neg B$

$\equiv F$

Example: Determine whether $\neg((A \Rightarrow B) \Rightarrow (A \wedge B))$ is a contingency, tautology, or contradiction.

$$\neg((A \Rightarrow B) \Rightarrow (A \wedge B))$$

① $\equiv \neg((\neg A \vee B) \Rightarrow (A \wedge B))$

$\equiv \neg(\neg(\neg A \vee B) \vee (A \wedge B))$

② $\equiv \neg\neg(\neg A \vee B) \wedge \neg(A \wedge B)$

$\equiv (\neg A \vee B) \wedge \neg(A \wedge B)$

$\equiv (\neg A \vee B) \wedge (\neg A \vee \neg B)$

$\equiv \neg A \vee (B \wedge \neg B)$

$\equiv \neg A \vee F$

$\equiv \neg A$

Therefore, $\neg((A \Rightarrow B) \Rightarrow (A \wedge B))$ is a contingency.

Propositional Functions

A proposition is a statement that is either true or false.

"$x < 5$"? Not a proposition, because the value of x was not given.

This is a <u>propositional function</u>, or <u>predicate</u>.

$P(x) =$ "$x < 5$" $P(x)$ becomes a proposition once we give a value to x.

$P(3)$ is true. $P(5)$ is false.

$P(\text{"monkey"}) = \text{"monkey} < 5\text{"}$ is nonsensical.

Every propositional function has a <u>universe of discourse</u> or <u>domain</u>. These are all the values that the variable can take.

For example, we can pick the universe of discourse for $P(x)$ to be $\{3, 4, 5, 6\}$.

<u>Quantifiers</u>

Suppose we want to say that $P(x)$ holds for every element of its universe of discourse:

$$P(3) \land P(4) \land P(5) \land P(6)$$

We can write this more succinctly:

$$\underbrace{\forall x\,(P(x))}_{\text{This is a proposition}} : \text{"For all } x,\ P(x)\text{"}$$

Truth value $\begin{cases} T & \text{if } P(x) \text{ is true for every } x \\ F & \text{if there is an } x \text{ for which } P(x) \text{ is false} \end{cases}$

(we call such an x a <u>counterexample</u>)

This is called a <u>universal</u> quantifier.

If, instead, we want to express that $P(x)$ holds for some element, ~~we~~

$$P(3) \vee P(4) \vee P(5) \vee P(6)$$

we can shorten this by using the <u>existential</u> quantifier:

$\exists x (P(x))$: "There exists an x such that $P(x)$"

Truth value $\begin{cases} T & \text{if there exists an } x \text{ for which } P(x) \text{ is true} \\ F & \text{if } P(x) \text{ is false for every } x. \end{cases}$

Example: $P(x) =$ "$x > 5$" UoD = all real numbers

$\forall x\, P(x)$ is fasle, because $P(\pi)$ is false

$\exists x\, P(x)$ is true, because $P(2\pi)$ is true

Example: $P(x) =$ "$x^2 \geqslant x$" UoD = all real numbers

$\forall x\, P(x)$ is false, because $P(1/2)$ is false.

If the UoD = all integers, then $\forall x\, P(x)$ is true.

~~Ex~~ Two statements involving quantifiers are logically equivalent if they have the same truth value, no matter which predicates are used and no matter which UoD is used.

$$\forall_x (P(x) \wedge Q(x)) \equiv (\forall_x P(x)) \wedge (\forall_x Q(x))$$

$$\cancel{\forall_x (P(x) \vee Q(x)) \not\equiv \forall_x P(x) \vee \forall_x Q(x)}$$

$$(P(x_1) \wedge Q(x_1)) \wedge (P(x_2) \wedge Q(x_2)) \wedge \ldots$$
$$\equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge Q(x_1) \wedge Q(x_2) \wedge \ldots$$

$$\exists_x (P(x) \vee Q(x)) \equiv \exists_x P(x) \vee \exists_x Q(x)$$

Note: $\forall_x (P(x) \vee Q(x)) \not\equiv (\forall_x P(x)) \vee (\forall_x Q(x))$

(Example: $P(x)$ = "x is even", $Q(x)$ = "x is odd", UoD = $\mathbb{Z}$)

Note: $\underbrace{\forall_x P(x)}_{\text{a proposition}} \wedge \underbrace{Q(x)}_{\text{meaningless: } x \text{ is \underline{unbound}}}$

Instead, write $\forall_x (P(x) \wedge Q(x))$

~~Note: $\exists_x P(x) \wedge \exists_x$~~

De Morgan

$$\neg(\forall_x P(x)) \equiv \exists_x (\neg P(x))$$
$$\neg(\exists_x P(x)) \equiv \forall_x (\neg P(x))$$

Example: $R(x)$ = "x is rich" UoD = all people

"Everyone is rich" = $\forall_x R(x)$

"Not everyone is rich" = $\neg(\forall_x R(x))$

$\equiv$

"Some one is not rich" = $\exists_x (\neg R(x))$

Example: $H(x)$ = "x is happy"

"All rich people are happy" $= \forall x\,(R(x) \Rightarrow H(x))$

"Not all rich people are happy"

$$= \neg(\forall x\,(R(x) \Rightarrow H(x)))$$
$$\equiv \exists x\,(\neg(R(x) \Rightarrow H(x)))$$
$$\equiv \exists x\,(\neg(\neg R(x) \vee H(x)))$$
$$\equiv \exists x\,(\neg\neg R(x) \wedge \neg H(x))$$
$$\equiv \exists x\,(R(x) \wedge \neg H(x))$$

= "There exists ~~someone~~ a rich person who is not happy."

Nesting Quantifiers

L3

Predicates can have more than one variable:

$$P(x,y) = x < y$$

In these cases, we often use nested quantifiers:

$$\underbrace{\forall x\,(\exists y\; P(x,y))}_{\text{true if UoD}=\mathbb{R},\ \text{false if UoD}=\mathbb{Z}}$$ = "For ~~all~~ each x, there exists a value y such that $x < y$."

Note: $\forall x \exists y\, P(x,y) \not\equiv \exists y \forall x\, P(x,y)$

Order of the <u>same</u> quantifiers doesn't matter:

$$\forall x \forall y\; P(x,y) \equiv \forall y \forall x\; P(x,y)$$