Example: Prove that $\sqrt{2}$ is irrational.

Proof: Assume that $\sqrt{2}$ is rational.

Then we can write $\sqrt{2} = \frac{a}{b}$ for two integers a and b, with $b \neq 0$, and such that a and b are not both even.

$$\sqrt{2} = \frac{a}{b}, \quad 2 = \frac{a^2}{b^2}, \quad a^2 = 2b^2$$

Therefore a^2 is even, which means that a is even.

So we can write $a = 2k$, for some integer k.

Then $a^2 = (2k)^2 = 4k^2 = 2b^2$, and $b^2 = 2k^2$.

Therefore b^2 is even, meaning that b is also even.

Thus, a and b are both even. $\lightning$

Conclusion: $\sqrt{2}$ is not rational.

L4

Example: If $n = ab$, where $a \geq 1$ and $b \geq 1$ are integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Proof: Assume $n = ab$, but $\neg(a \leq \sqrt{n} \vee b \leq \sqrt{n})$.

So $a > \sqrt{n}$ and $b > \sqrt{n}$. But then ~~ab~~

$$n = ab > \sqrt{n} \cdot \sqrt{n} = n. \quad \lightning$$

If we need to prove $(P_1 \vee P_2 \vee \dots \vee P_k) \Rightarrow C$, we can instead prove $(P_1 \Rightarrow C) \wedge (P_2 \Rightarrow C) \wedge \dots \wedge (P_k \Rightarrow C)$.

This is a proof by cases.

Example: Prove that $|xy| = |x||y|$ for all real numbers x, y.

Proof: Consider the following cases:

1) $x \geq 0$ and $y \geq 0$ then
2) $x \geq 0$ and $y < 0$ ...
3) $x < 0$ and $y \geq 0$ ...
4) $x < 0$ and $y < 0$ ...

Since these cases cover all possibilities, and the statement is true in each case, the statement holds for all real numbers.

If we need to prove an equivalence, or if-and-only-if, $P \Leftrightarrow C$, we must prove both $P \Rightarrow C$ and $C \Rightarrow P$.

Example: pq is even iff p is even or q is even

1) If pq is even, then p is even or q is even

Indirect proof: assume $\neg$(p is even or q is even)
Then p is odd and q is odd, so $p = 2k+1$ and $q = 2x+1$ for some integers k, x.

$$pq = (2k+1)(2x+1) = 4kx + 2k + 2x + 1 = 2(2kx + k + x) + 1$$

Thus, pq is odd.

2) If p is even or q is even, then pq is even.

Sets

- A set is an unordered collection of things.

  $S = \{a, c, 5, x\}$

  Order does not matter: $\{a, c, 5, x\} = \{c, x, 5, a\}$

  Repetition does not matter: $\{a, c, 5, x\} = \{a, a, 5, c, 5, x, a\}$

  Sets can contain sets: $\{a, c, \{a, c\}, 5, \{5\}\}$

- Objects in the set are called elements of the set.

  $a \in S$: "a is an element of S"

  $a \notin S$: "a is not an element of S"

- We often define sets like this:

  $\{x \mid x \text{ is an integer and } 1 \leq x \leq 20\}$

  $= \{1, 2, 3, 4, \ldots, 20\}$

  Read as "The set of all x such that ..."

- Important sets:

  $\mathbb{N}$: natural numbers : $\{0, 1, 2, 3, \ldots\}$

  $\mathbb{Z}$: integers : $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

  $\mathbb{Q}$: rational numbers : $\{p/q \mid p \in \mathbb{Z} \wedge q \in \mathbb{Z} \wedge q \neq 0\}$

  $\mathbb{R}$: real numbers

  $\emptyset$: empty set : $\{\}$

- The cardinality of S is the number of distinct elements of S:

  $S = \{a, c, 5, x\}$ $\quad |S| = 4$ $\qquad |\mathbb{N}| = \infty$

  $|\{x \mid x \in \mathbb{N} \wedge x < 50\}| = 50$

  $|\emptyset| = 0$

- Two sets, A and B, are <u>equal</u> if they contain exactly the same elements:

$$A = B \equiv \forall x (x \in A \Leftrightarrow x \in B)$$

- A is a <u>subset</u> of B if each element of A is also an element of B:

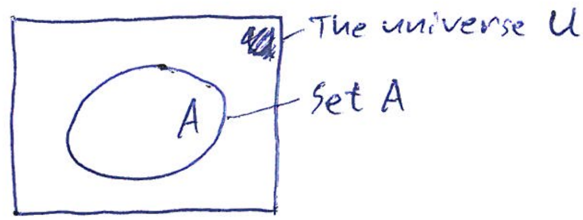
$$A \subseteq B \equiv \forall x (x \in A \Rightarrow x \in B)$$

Examples:

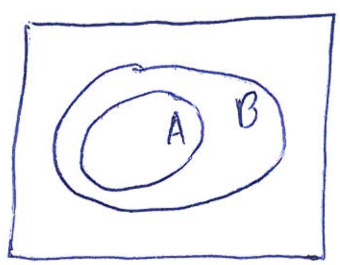
$\{a, x\} \subseteq \{a, c, 5, x\}$ $\{5, a\} \subseteq \{a, c, 5, x\}$

$\mathbb{N} \subseteq \mathbb{Z}$, $\mathbb{Z} \subseteq \mathbb{Q}$, $\mathbb{Q} \subseteq \mathbb{R}$

<u>Venn Diagram:</u>



$A \subseteq B$



- We can prove things about sets:

<u>Claim:</u> For any set S, $\emptyset \subseteq S$.

<u>Proof:</u> We need to show that $\forall x (x \in \emptyset \Rightarrow x \in S)$.

The empty set has no elements, so $x \in \emptyset$ is False for all x. Since "$x \in \emptyset$" is False, "$x \in \emptyset \Rightarrow x \in S$" is True for all x. Therefore $\emptyset \subseteq S$. $\square$

~~t. These~~

- A is a <u>proper subset</u> of B if $A \subseteq B$ and $A \neq B$:

$$A \subset B \equiv \forall x (x \in A \Rightarrow x \in B) \wedge \exists x (x \in B \wedge x \notin A)$$

- The <u>power set</u> of a set S is the set of all subsets of S:

Not in L4

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}$$

<u>Example</u>: $S = \{1, 2, 3\}$

$$\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Note: $|\mathcal{P}(S)| = 8 = 2^3 = 2^{|S|}$

Note: $A \in \mathcal{P}(S) \equiv A \subseteq S$

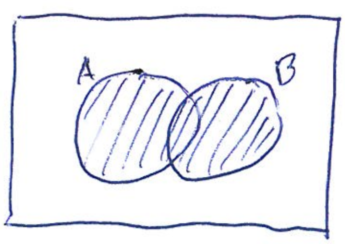
$\emptyset \in \mathcal{P}(S)$, $S \in \mathcal{P}(S)$

<u>Set operations</u>

We can combine sets in ~~differe~~ several ways.

- <u>Union</u> of A and B:

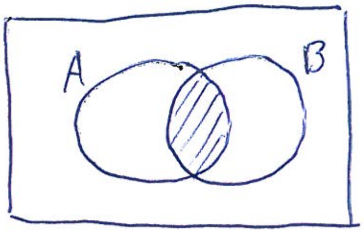
$$A \cup B = \{x \mid x \in A \vee x \in B\}$$



$$\{a, c, 5, x\} \cup \{3, 5, a\} = \{a, c, 5, x, 3\}$$

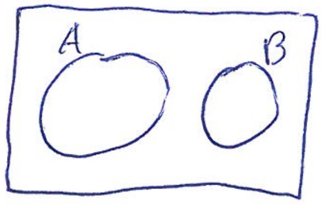
- <u>Intersection</u> of A and B:

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$



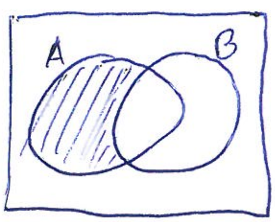
$$\{1,3,5\} \cap \{1,2,3\} = \{1,3\}$$

- We say A and B are <u>disjoint</u> if $A \cap B = \emptyset$.



- <u>Difference</u> of A and B:

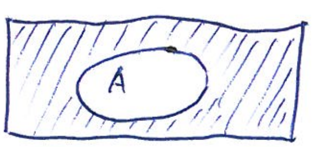
$$A - B = \{x \mid x \in A \wedge x \notin B\}$$



$$\{1,3,5\} - \{1,2,3\} = \{5\}$$

- <u>Complement</u> of A:

$$\overline{A} = \{x \mid x \notin A\}$$



Note: $\overline{A} = U - A$

Set Identities

Remember: $A = B \equiv \forall x (x \in A \Leftrightarrow x \in B)$

To prove that $A=B$ ~~we typically~~ it is sometimes easier to show that

$\underbrace{A \subseteq B}_{\forall x (x \in A \Rightarrow x \in B)}$ and $\underbrace{B \subseteq A}_{\forall x (x \in B \Rightarrow x \in A)}$.

(* do 29 first)

$A \cap U = A$ $\qquad A \cup \emptyset = A$

$A \cap \emptyset = \emptyset$ $\qquad A \cup U = U$

$A \cap A = A$ $\qquad A \cup A = A$

$A \cap \overline{A} = \emptyset$ $\qquad A \cup \overline{A} = U$

$\overline{\overline{A}} = A$

$A \cap B = B \cap A$ $\qquad A \cup B = B \cup A$

$(A \cap B) \cap C = A \cap (B \cap C)$ $\qquad (A \cup B) \cup C = A \cup (B \cup C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $\qquad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\qquad \overline{A \cup B} = \overline{A} \cap \overline{B}$

$A \cap (A \cup B) = A$ $\qquad A \cup (A \cap B) = A$

$A - B = A \cap \overline{B}$ (Difference Equivalence)

Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

~~Proof:~~

~~$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$: $\forall x (x \in \overline{A \cap B} \Rightarrow x \in \overline{A} \cup \overline{B})$~~

~~Let $x \in \overline{A \cap B}$. Then $x \notin A \cap B$, which means $\neg(x \in A \wedge x \in B)$.~~

$$\cancel{\begin{aligned} &\neg(x \in A \wedge x \in B) \\ &\equiv \neg(x \in A) \vee \neg(x \in B) \\ &\equiv x \in \overline{A} \vee x \in \overline{B} \\ &\equiv x \in \overline{A} \cup \overline{B} \end{aligned}}$$

Proof:
$$\begin{aligned} \overline{A \cap B} &= \{x \mid x \in \overline{A \cap B}\} \\ &= \{x \mid x \notin A \cap B\} \\ &= \{x \mid \neg(x \in A \cap B)\} \\ &= \{x \mid \neg(x \in A \wedge x \in B)\} \\ &= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} \\ &= \{x \mid x \notin A \vee x \notin B\} \\ &= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} \\ &= \{x \mid x \in \overline{A} \cup \overline{B}\} \\ &= \overline{A} \cup \overline{B} \end{aligned}$$

Claim: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof:
$$\begin{aligned} A \cap (B \cup C) &= \{x \mid x \in A \wedge x \in B \cup C\} \\ &= \{x \mid x \in A \wedge (x \in B \vee x \in C)\} \\ &= \{x \mid (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\} \\ &= \{x \mid x \in A \cap B \vee x \in A \cap C\} \\ &= \{x \mid x \in (A \cap B) \cup (A \cap C)\} \\ &= (A \cap B) \cup (A \cap C) \end{aligned}$$

Example: Is $(A-C)\cap(B-C)=(A\cap B)\cap\overline{C}$?

Proof: $(A-C)\cap(B-C) = (A\cap\overline{C})\cap(B\cap\overline{C})$
$= (A\cap B)\cap(\overline{C}\cap\overline{C})$
$= (A\cap B)\cap\overline{C}$

L5

Tuples

In a set, order of elements doesn't matter. So how do we represent ~~things~~ collections where order *does* matter?

Ordered n-tuple: $(a_1, a_2, a_3, \ldots, a_n)$

If $n=2$, ordered pair; $n=3$, ordered triple.

Equality: $(a_1, a_2, a_3, \ldots, a_n) = (b_1, b_2, b_3, \ldots, b_n)$ if
$a_1=b_1 \wedge a_2=b_2 \wedge a_3=b_3 \wedge \ldots \wedge a_n=b_n$.

Cartesian Product of sets A and B:

$$A\times B = \{(a,b) \mid a\in A \wedge b\in B\}$$

Example: $\{1,2\}\times\{a,b,c\} = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)\}$
$\{a,b,c\}\times\{1,2\} = \{(a,1),(a,2),(b,1),(b,2),(c,1),(c,2)\}$

~~Note:~~ In general, $A\times B \neq B\times A$

$\mathbb{R}\times\mathbb{R} = \{(x,y) \mid x\in\mathbb{R} \wedge y\in\mathbb{R}\}$ = plane

We sometimes write $\mathbb{R}^2$.