Vacuous Truth

$\boxed{\text{Teaching Evals}}$

$\boxed{\text{L14}}$ (review)

$\text{False} \Rightarrow P \equiv \text{True}$, regardless of P.

Consequence:

$$\begin{aligned}
\forall x \in \emptyset (P(x)) &\equiv \forall x (x \in \emptyset \Rightarrow P(x)) \\
&\equiv \forall x (\text{False} \Rightarrow P(x)) \\
&\equiv \forall x (\text{True}) \\
&\equiv \text{True}
\end{aligned}$$

Regardless of $P(x)$!

Example: "All unicorns in this room are pink."

This happens in the base case of induction:

$\forall x \in S (P(x))$. Base Case: $|S| = 0$ ✓.

This happens with relations:

$\overset{a}{\cdot} \quad \overset{b}{\cdot} \qquad R = \emptyset$

$\forall a,b \, ((a,b) \in R \Rightarrow (b,a) \in R)$ ✓

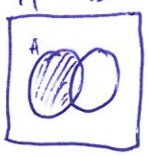
Sets

Example: Is the following valid?

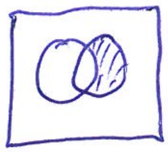
$$(A-B)\cup(B-A)=(A\cap B)$$

Solution: Draw the Venn diagram:

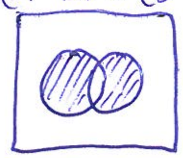
$A-B$



$B-A$

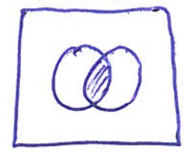


$(A-B)\cup(B-A)$



$\neq$

$A\cap B$



~~Proof, that (A-B)∪(B-A)~~

So consider ~~a~~ sets $A=\{1\}$ and $B=\emptyset$.

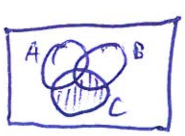
$$\begin{aligned}(A-B)\cup(B-A)&=(\{1\}-\emptyset)\cup(\emptyset-\{1\})\\&=\{1\}\cup\emptyset\\&=\{1\}\end{aligned}$$

$$A\cap B=\emptyset\neq\{1\}.$$

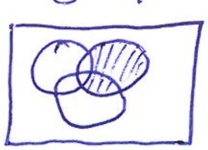
Example: Is "$(C-A)\cap(B-A)=(C\cap B)-A$" valid?

Solution: Draw:

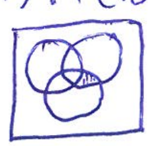
$C-A$



$B-A$

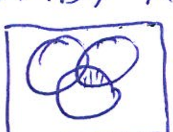


$(C-A)\cap(B-A)$



$C\cap B$

$(C\cap B)-A$



To show this:

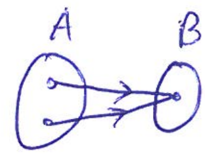
$$
\begin{aligned}
(C-A)\cap(B-A) &= (C\cap\bar{A})\cap(B\cap\bar{A}) && \text{(Difference equiv.)}\\
&= (C\cap B)\cap(\bar{A}\cap\bar{A}) && \text{(Comm. \& Assoc.)}\\
&= (C\cap B)\cap\bar{A} && \text{(Idempotence)}\\
&= (C\cap B)-A && \text{(Difference Equiv.)}
\end{aligned}
$$

Properties of Functions

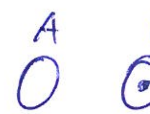
Function $f: A \to B$.

Injective: $\forall x,y \in A\,(f(x)=f(y) \Rightarrow x=y)$

or $\forall x,y \in A\,(x \neq y \Rightarrow f(x) \neq f(y))$

"All pre-images are unique." No 

Surjective: $\forall b \in B\,(\exists a \in A\,(f(a)=b))$

"Everyone has a pre-image." No 

Bijective: Both injective and surjective.

"Everyone has a unique pre-image."

Example: $f: \mathbb{N} \to \mathbb{N}$, $f(x) = x+1$.

f is injective:
$$
\begin{aligned}
f(x) &= f(y) \\
x+1 &= y+1 \\
x &= y
\end{aligned}
$$

f is not surjective: there is no x such that $f(x) = 0$.

f is not bijective: f is not surjective.

Example: $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = x+1$

f is injective:
$$
\begin{aligned}
f(x) &= f(y) \\
x+1 &= y+1 \\
x &= y
\end{aligned}
$$

f is surjective:
$$
\begin{aligned}
f(x) &= b \\
x+1 &= b \\
x &= b-1 \qquad b \in \mathbb{Z} \Rightarrow (b-1) \in \mathbb{Z}.
\end{aligned}
$$

f is bijective.

Example: $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$

f is not injective: $f(-1) = f(1)$

f is not surjective: there is no $x \in \mathbb{Z}$ such that $f(x) = -1$.

f is not bijective.

Induction

- $P(n)$: predicate on natural numbers
  (Can be complex: "All graphs with n vertices have ...
- Prove $P(n)$ for all $n \geq a$.

Base Case:

- ~~P(a) is true~~ prove $P(a)$
- ~~P(a+1) is true~~ prove $P(a+1)$

  $\vdots$
- ~~P(b) is true~~ prove $P(b)$

Inductive Hypothesis:

Assume $P(n)$ is true for $a \leq n \leq k$, for some $k \geq b$.

Inductive Step: Show that $P(k+1)$ is true.

Example: Prove that, for $n \geq 5$, $4n < 2^n$.

Proof: By induction on n.

Base Case ($n=5$): $4 \cdot 5 = 20$ $\quad 20 < 32$ ✓

$2^5 = 32$

Inductive Hypothesis: Assume that $4n < 2^n$ for ~~all~~ $5 \leq n \leq k$.

Inductive Step: We need to show that $4 \cdot (k+1) < 2^{k+1}$.

$$
\begin{aligned}
4(k+1) &= 4k + 4 \\
&< 2^k + 4 && \text{(IH)} \\
&< 2^k + 2^k && (k \geq 5) \\
&= 2^{k+1}
\end{aligned}
$$
✓

Equivalence Relations

An equivalence relation is reflexive, symmetric and transitive.

Example: Show that the relation on $\mathbb{R}$ given by
$R = \{(a,b) \mid a-b \in \mathbb{Z}\}$ is an equiv. relation.

Solution:

- R is reflexive: $a-a=0$, and $0 \in \mathbb{Z}$, so $(a,a) \in R$ for all $a \in \mathbb{R}$.

- R is symmetric: Suppose $a,b \in \mathbb{R}$ and $a-b \in \mathbb{Z}$.
Then $a-b=k$ and $b-a=-(a-b)=-k$.
If $k \in \mathbb{Z}$, then $-k \in \mathbb{Z}$, so $(b,a) \in R$.

- R is transitive: Suppose $(a,b) \in R$ and $(b,c) \in R$.
Then $a-b=k \in \mathbb{Z}$ ~~and $b-c=\ell \in \mathbb{Z}$~~
~~So $a-b+(b-c)=$~~

$$
\begin{aligned}
& \quad a-b=k \in \mathbb{Z} \\
\text{and } & \quad \underline{b-c=\ell \in \mathbb{Z}} \; + \\
\text{so } & \quad a-c=(k+\ell) \in \mathbb{Z}.
\end{aligned}
$$