Example: Is $(A-C)\cap(\cancel{A\cap}B-C)=(A\cap B)\cap\bar{C}$?

Proof:
$$\begin{aligned}(A-C)\cap(B-C) &= (A\cap\bar{C})\cap(B\cap\bar{C})\\ &=(A\cap B)\cap(\bar{C}\cap\bar{C})\\ &=(A\cap B)\cap\bar{C}\end{aligned}$$

L5

Tuples

In a set, order of elements doesn't matter. So how do we represent ~~things~~ collections where order does matter?

Ordered n-tuple: $(a_1, a_2, a_3, \ldots, a_n)$

If $n=2$, ordered pair; $n=3$, ordered triple.

Equality: $(a_1, a_2, a_3, \ldots, a_n) = (b_1, b_2, b_3, \ldots, b_n)$ if
$a_1=b_1 \wedge a_2=b_2 \wedge a_3=b_3 \wedge \ldots \wedge a_n=b_n$.

Cartesian Product of sets A and B:

$$A\times B=\{(a,b)\mid a\in A\wedge b\in B\}$$

Example:
$\{1,2\}\times\{a,b,c\}=\{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)\}$
$\{a,b,c\}\times\{1,2\}=\{(a,1),(a,2),(b,1),(b,2),(c,1),(c,2)\}$

~~Note:~~ In general, $A\times B\neq B\times A$

$\mathbb{R}\times\mathbb{R}=\{(x,y)\mid x\in\mathbb{R}\wedge y\in\mathbb{R}\}$ = plane

We sometimes write $\mathbb{R}^2$.

$$A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_1 \in A_1 \wedge a_2 \in A_2 \wedge \dots \wedge a_n \in A_n\}$$

Exercise: Show that $A \times \emptyset = \emptyset$.

Proof: ~~$A\times\emptyset = \{(a,b)\mid$~~

$A \times \emptyset \subseteq \emptyset$: $\forall x (x \in A \times \emptyset \Rightarrow x \in \emptyset)$

Let $x \in A\times\emptyset$. Then $x = (a,b)$ with $a \in A$ and $b \in \emptyset$.
But $b \in \emptyset$ is False. Therefore $x \in A \times \emptyset$ is False,
and $x \in A\times\emptyset \Rightarrow x \in \emptyset$ is True.

$\emptyset \subseteq A \times \emptyset$: $\forall x (x \in \cancel{A\times}\emptyset \Rightarrow x \in A \times \emptyset)$

~~Let~~ $x \in \emptyset$ is False for all x. Therefore
$x \in \emptyset \Rightarrow x \in A\times\emptyset$ is True.

Functions

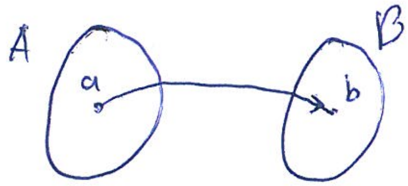
In highschool: $f(x) = x^2 + 11$

this means : each element $x \in \mathbb{R}$ is ~~assigned~~ mapped to the real number x^2+11.

More general: A and B are non-empty sets.

Function f from A to B:

Every element of A is mapped to exactly one element of B.



$f: A \to B$

$f(a) = b$

Example:

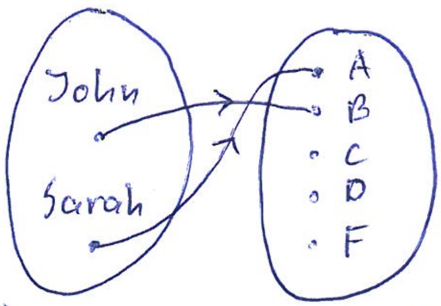
$A = \{John, Sarah\}$ $\quad f: A \to B$

$B = \{F, D, C, B, A\}$

$f(John) = B$

$f(Sarah) = A$

Enumeration



Graphical representation

$f = \{(John, B), (Sarah, A)\}$

$f \subseteq A \times B$

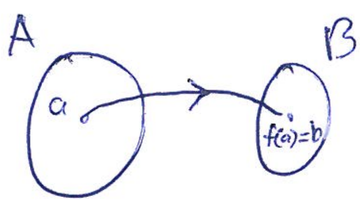
Set representation

Terminology

Let $f: A \to B$.

A: domain of f $\qquad$ f maps A to B

B: codomain of f

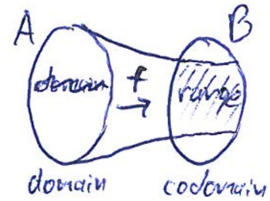


b: image of a

a: pre-image of b

range of f : set of all images of f

$= \{f(a) \mid a \in A\} = \{b \mid \exists a (a \in A \wedge b = f(a))\}$



Example: $f: \underbrace{\mathbb{N}}_{\text{domain}} \to \underbrace{\mathbb{N}}_{\text{codomain}}$, $f(x) = x^2$

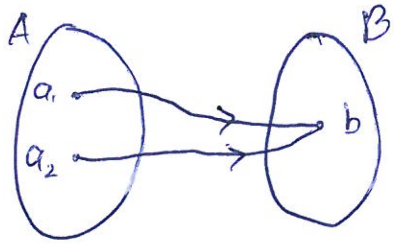
range $= \{0, 1, 4, 9, 16, 25, \ldots\}$

Properties of functions

- $f: A \to B$ is <u>injective</u> (one-to-one) if

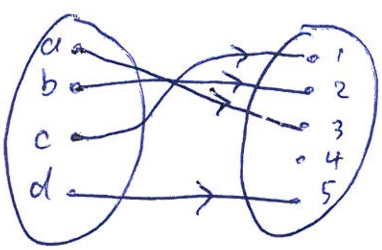
$$\forall a, b \in A \, (f(a) = f(b) \Rightarrow a = b)$$
$$\equiv$$
$$a \neq b \Rightarrow f(a) \neq f(b)$$

Essentially, this does not happen:



Example

$f: \{a, b, c, d\} \to \{1, 2, 3, 4, 5\}$



f is injective

$g: \mathbb{R} \to \mathbb{R}, \; g(x) = x^2$

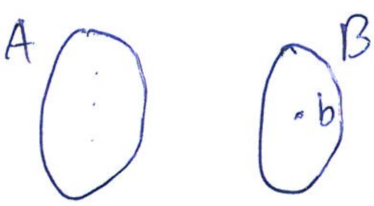
$g(1) = 1$
$g(-1) = 1$

g is not injective

- $f: A \to B$ is <u>surjective</u> (onto) if the range = codomain.

$$\forall b \in B \, (\exists a \in A \, (f(a) = b))$$

This does not happen:



- $f: A \to B$ is <u>bijective</u> if it is <u>both</u> injective and surjective.

To prove that a function is:

- injective: assume $f(x) = f(y)$ and show that $x = y$.
- not injective: find x and y such that $x \neq y$, but $f(x) = f(y)$
- surjective: take an arbitrary element $b \in B$ and show an $a \in A$ such that $f(a) = b$.
- not surjective: find a $b \in B$ such that $f(a) \neq b$ for any $a \in A$
- bijective: show that it is both injective and surjective

Example:

$f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = 3x + 2$

- injective:

$$
\begin{aligned}
f(x) &= f(y) \\
3x + 2 &= 3y + 2 \\
3x &= 3y \\
x &= y \quad \square
\end{aligned}
$$

- not surjective:

$$
\begin{aligned}
f(x) &= 4 \\
3x + 2 &= 4 \\
3x &= 2 \\
x &= 2/3 \notin \mathbb{Z} \quad \square
\end{aligned}
$$

- not bijective: not surjective.

Example: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 3x+2$

- injective: same as before.
- surjective:

$$\begin{aligned} f(x) &= y \\ 3x+2 &= y \\ 3x &= y-2 \\ x &= \frac{y-2}{3} \in \mathbb{R} \quad \square \end{aligned}$$

- bijective: injective and surjective.

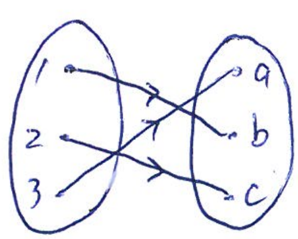
The *inverse* of a bijection $f: A \to B$ is the function $f^{-1}: B \to A$ defined as follows.

Let $b \in B$. Since f is a bijection, there is a *unique* element $a \in A$ such that $f(a) = b$. We define $f^{-1}(b) = a$.

Example: The inverse of $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 3x+2$ is $f^{-1}(y) = (y-2)/3$.

Example: $f: \{1,2,3\} \to \{a,b,c\}$

$f(1) = b$
$f(2) = c$
$f(3) = a$



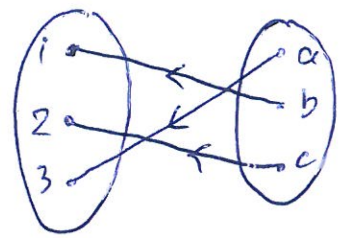
f is a bijection.

$f^{-1}: \{a,b,c\} \to \{1,2,3\}$: reverse the arrows

$f^{-1}(a) = 3$

$f^{-1}(b) = 1$

$f^{-1}(c) = 2$

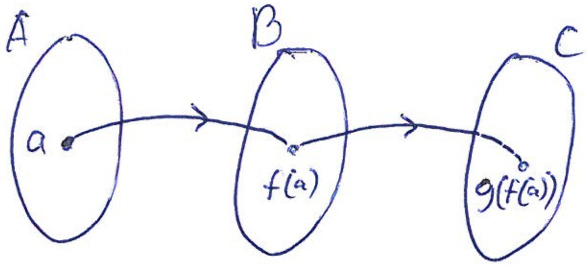


Function Composition

Given functions $f: A \to B$ and $g: B \to C$, we can define a new function $\underline{g \circ f}: A \to C$.

"g after f"

$$(g \circ f)(a) = g(f(a))$$



Example: ~~Let~~ $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 3x^2 - 5x + 17$

$g: \mathbb{R} \to \mathbb{R}$, $g(x) = 2x + 6$

$$\begin{aligned} g \circ f: \quad (g \circ f)(x) &= g(f(x)) \\ &= 2(f(x)) + 6 \\ &= 2(3x^2 - 5x + 17) + 6 \\ &= 6x^2 - 10x + 40 \end{aligned}$$

$$\begin{aligned} f \circ g: \quad (f \circ g)(x) &= f(g(x)) \\ &= f(2x+6) \\ &= 3(2x+6)^2 - 5(2x+6) + 17 \\ &= 3(4x^2 + 24x + 36) - 5(2x+6) + 17 \\ &= 12x^2 + 62x + 95 \end{aligned}$$

Note: $g \circ f \neq f \circ g$